*International Journal of Theoretical Physics, Vol. 45, No. 2, February 2006 (© 2006)*
DOI: 10.1007/s10773-006-9028-0

# Comoving Self-Gravitating Scalar Field in the Newman Penrose Formalism

**Antonio Zecca**[1,2]

*Received February 3, 2005; accepted January 19, 2006*
*Published Online: April 27, 2006*

The Einstein field equation, coupled to the scalar field, is studied in a spherically symmetric comoving system. The problem is translated into the language of the Newman Penrose formalism that is based on the choice of a null tetrad frame. The corresponding (tabulated) Einstein field equation, Bianchi identities and scalar field equation are explicited in terms of the Weyl and Ricci scalars and discussed. Spherical symmetry reduces the difficulties but not so far to enable to integrate the scheme in general. The main result is that static self-gravitation is possible only for massless scalar field. The static solution is determined. It depends on an arbitrary function that can be interpreted as radial coordinate. The part of the space–time solution of the problem does not contain black holes. It is remarked that in the part of the space–time not solution of the problem, light rays cannot propagate radially but admit circular orbits.

**KEY WORDS:** self-gravitation; Newman Penrose formalism; trapped surfaces; black hole.

## 1. INTRODUCTION

The study of self-gravitational interaction of fields has received great attention in the literature mainly with regard to the simplest case represented by the scalar field. The interest is connected also to a better understanding of cosmological models, gravitational collapse and black holes formation (Krasinski, 1997; Weinberg, 1972). In case of spherically symmetric space–time with, a priori, two independent unknown functions it has indeed been shown that solutions exist for massless field that may lead to black holes formation (Christodoulou 1986, 1987). The result has been numerically characterized by a mass scaling exponent associated to the formation of black holes (Choptuik, 1993) and it has been extended to axisymmetric space–time (Abraham and Evans, 1993; Choptuik *et al*., 2003; Whang, 2003), to the collapse of fluid (Evans and Coleman, 1994) and to complex

[1] Dipartimento di Fisica dell' Universita' and INFN, Via Celoria 16, 20133 Milan, Italy; e-mail: Antonio.Zecca@mi.infn.it.
[2] Gruppo Nazionale par la Fisica Matematica, Italy

scalar field (Hirshman and Eardley, 1995a,b). The problem has been studied, in an elementary way, also in the context of a class of spherically symmetric comoving system, a situation that still depends, a priori, on two general functions. The corresponding case of static massless scalar field, whose solution depends on an arbitrary function, was studied in Zecca (2001). The cases of the a priori assumption of static massless or massive time-dependent scalar field was discussed in Zecca (2004). The attention in the two last mentioned papers was however essentially directed towards the determination of pure analytical solutions of the problem.

In the present paper, the self-gravitation of scalar field in spherically symmetric comoving system is studied in the language of the Newman Penrose formalism. Under the choice of a suitable null tetrad frame it is first shown that the symmetry of the problem implies the independence of the field from the angular coordinates. This easily follows by applying the Einstein equation and Bianchi identities in terms of the Weyl and Ricci "scalars," spin coefficients and directional derivatives and from some functional identities. The Newman Penrose formalism allows an alternative way in the solution of the coupled equations. The possibility is explicited for the completely static situation that has no solution for massive scalar field, while it is integrated in the massless case in agreement with previous results. The solution depends on an arbitrary function, with no physical significance, that can be re-interpreted as the radial coordinate.

The advantage of the Newman Penrose formalism is also of making here transparent many general aspects concerning light propagation. This is evidenced by considering the optical scalars equation (and similar equations for other geometrical objects) that are discussed here. In the static case, the space–time solution does not contain black holes. There are regions of the space–time that are not solution of the problem, where light rays cannot propagate radially, but where circular light orbits are possible.

## 2. THE EQUATIONS IN THE NEWMAN PENROSE FORMALISM

The coupled Einstein and scalar field equations describing the self-gravitation of the scalar field $\phi$ are expressed by

$$R_{\mu\nu} = -k\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_0^2\phi^2 g_{\mu\nu}\right) \tag{1}$$

$$\nabla^\alpha\nabla_\alpha\phi + m^2\phi = 0 \qquad (k = 8\pi G/c^4) \tag{2}$$

where $m_0$ is the mass of the particle of the scalar field, $\nabla_\alpha$ the covariant torsion-free derivative in the space–time of metric tensor $g_{\mu\nu}$. Equation (1) is the Einstein field equation having as a source the energy momentum tensor given by $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial^\alpha\phi\partial_\alpha\phi - m_0^2\phi^2)$. Equations (1) and (2) will be studied in the spherically symmetric comoving system, with the tensor metric defined by

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = dt^2 - e^{\Gamma(r,t)}\, dr^2 - Y^2(r,t)(d\theta^2 + \sin^2\theta\, d^2\varphi), \tag{3}$$

The energy momentum tensor is automatically conserved, $\nabla^\mu T_{\mu\nu} = 0$, as a consequence of the validity of the scalar field equation (e.g., Zecca, 2003). The scheme will be studied by adopting the Newman Penrose (1962) formalism. The null tetrad frame $e_a^\mu$ (the row Latin index run over the tetrad components 1, 2, 3, 4; the column Greek index over $t, r, \theta, \varphi$) is chosen to be the one originally considered in Zecca (1993):

$$e_a^\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{-\Gamma/2} & 0 & 0 \\ 1 & -e^{-\Gamma/2} & 0 & 0 \\ 0 & 0 & 1 & i\csc\theta/Y \\ 0 & 0 & 1 & -i\csc\theta/Y \end{pmatrix} \tag{4}$$

Correspondingly, the directional derivatives are defined by $D = e_1^\mu\partial_\mu$, $\Delta = e_2^\mu\partial_\mu$, $\delta = e_3^\mu\partial_\mu$, $\delta^\star = e_4^\mu\partial_\mu$, while the non-zero spin coefficients resulting from the earlier choice (for notations and mathematical conventions we follow Chandrasekhar, 1983) are

$$\begin{aligned} &\rho = -\tfrac{1}{Y\sqrt{2}}(\dot{Y} + Y' e^{-\Gamma/2}), \qquad &&\mu = \tfrac{1}{Y\sqrt{2}}(\dot{Y} - Y' e^{-\Gamma/2}) \\ &\beta = -\alpha = \tfrac{1}{Y\sqrt{2}}\cot\theta, &&\epsilon = -\gamma = \tfrac{1}{4\sqrt{2}}\dot{\Gamma} \end{aligned} \tag{5}$$

(prime and dot mean $\partial_r$, $\partial_t$ respectively).

The Einstein field equations can be studied in terms of the component of the Weyl tensor represented by the complex scalar as $\psi_k$, $k = 0, 1, 2, 3$, and in terms of the Ricci real and complex scalars. The definition of these scalars and their expressions obtained from assumptions (1) and (4), are

$$\begin{aligned} &\phi_{00} = -\frac{1}{2}R_{(1)(1)} = \frac{k}{2}(D\phi)^2, \qquad &&\phi_{22} = -\frac{1}{2}R_{(2)(2)} = \frac{k}{2}(\Delta\phi)^2 \\ &\phi_{02} = -\frac{1}{2}R_{(3)(3)} = \frac{k}{2}(\delta\phi)^2, &&\phi_{20} = -\frac{1}{2}R_{(4)(4)} = \frac{k}{2}(\delta^\star\phi)^2 \\ &\phi_{01} = -\frac{1}{2}R_{(1)(3)} = \frac{k}{2}\,\delta\phi\,\delta^\star\phi, &&\phi_{10} = -\frac{1}{2}R_{(1)(4)} = \frac{k}{2}\,D\phi\,\Delta\phi \\ &\phi_{12} = -\frac{1}{2}R_{(2)(3)} = \frac{k}{2}\,\Delta\phi\,\delta\phi, &&\phi_{21} = -\frac{1}{2}R_{(2)(4)} = \frac{k}{2}\,\Delta\phi\,\delta^\star\phi \\ &\phi_{11} = -\frac{1}{2}\left(R_{(1)(2)} + R_{(3)(4)}\right) = \frac{k}{4}(D\phi\Delta\phi + \delta\phi\delta^\star\phi) && \\ &\Lambda = \frac{1}{12}\left(R_{(1)(2)} - R_{(3)(4)}\right) = -\frac{k}{12}\left(D\phi\Delta\phi - \delta\phi\,\delta^\star\phi - m_0^2\phi^2\right) && \end{aligned} \tag{6}$$

The definition $R_{ab} = R_{\mu\nu}\, e_a^\mu e_b^\nu$ has been used. If now one considers the Einstein field equation in terms of the directional derivatives, spin coefficients, Weyl and

Ricci scalars (Chandrasekhar, 1983; Newman and Penrose, 1962; Penrose and Rindler, 1984) one immediately finds in the present scheme:

$$\begin{array}{ll} \delta\phi = \delta^\star\phi = 0 & \Longrightarrow \quad \phi = \phi(r,t) \\ \psi_0 = \psi_1 = \psi_3 = \psi_4 = 0, & \psi_2 = \psi_2(r,t). \end{array} \tag{7}$$

As a consequence the expressions (6) greatly simplify. The non-trivial ones are:

$$\begin{array}{ll} \phi_{00} = \frac{k}{2}(D\phi)^2, & \phi_{22} = \frac{k}{2}(\Delta\phi)^2 \\ \phi_{11} = \frac{k}{4}D\phi\Delta\phi, & \Lambda = \frac{k}{12}(m_0^2\phi^2 - D\phi\Delta\phi) \end{array} \tag{8}$$

From the results it is clear that, as it is known (e.g. Krasinski, 1997; Zecca, 1993, 2000), the Weyl tensor results to be of Petrov type D (e.g. Chandrasekhar, 1983). The surviving equations are then:

$$D\rho = \rho^2 + 2\epsilon\rho + \phi_{00} \tag{9}$$

$$(D+\Delta)\gamma = -4\gamma\epsilon + \phi_{11} - \Lambda + \psi_2 \tag{10}$$

$$D\mu = \rho\mu - 2\mu\epsilon + 2\Lambda + \psi_2 \tag{11}$$

$$(\delta + \delta^\star)\alpha = \mu\rho + 4\alpha^2 + \phi_{11} + \Lambda - \psi_2 \tag{12}$$

$$\Delta\mu = -\mu^2 - 2\mu\gamma - \phi_{22} \tag{13}$$

$$\Delta\rho = -\rho\mu + 2\gamma\rho - 2\Lambda - \psi_2 \tag{14}$$

$$D\psi_2 - 3\rho\psi_2 + \Delta\phi_{00} - 2\rho\phi_{11} + (\mu - 4\gamma)\phi_{00} + 2D\Lambda = 0 \tag{15}$$

$$\Delta\psi_2 + 3\mu\psi_2 + D\phi_{22} + 2\mu\phi_{11} - (\rho - 4\epsilon)\phi_{22} + 2\Delta\Lambda = 0 \tag{16}$$

$$D(\phi_{11} + 3\Lambda) + \Delta\phi_{00} = 4\rho\phi_{11} - (2\mu - 4\gamma)\phi_{00} \tag{17}$$

$$\Delta(\phi_{11} + 3\Lambda) + D\phi_{22} = -4\mu\phi_{11} + (2\rho + 4\gamma)\phi_{22}. \tag{18}$$

Equations (9)–(14) are Einstein equations, while Eqs. (15)–(18) are Bianchi identities. The other Einstein equations, Bianchi identities and "eliminant relations" (Chandrasekhar, 1983) are identically satisfied or are function of the equations listed earlier and/or consequence of the following identities

$$\begin{array}{ll} DY = -\rho\, Y, & \Delta\, Y = \mu\, Y \\ D\,\beta = \rho\,\beta, & \Delta\,\beta = \beta\mu. \end{array} \tag{19}$$

that can be easily proved to hold.

*Remark.* One can partially check the correctness of the Einstein equations. By summing and subtracting the equation obtained from the sum of Eqs. (9) and (14) with the one obtained by summing Eqs. (11) and (13), and by using the explicit expressions of $\rho,\ \mu,\ \Lambda,\ \phi_{00},\ \psi_2$ one obtains

$$k\,\dot\phi\,\phi' = 2\frac{\dot Y'}{Y} - \frac{Y'}{Y}\dot\Gamma$$

$$\psi_2 = \frac{\ddot{Y}}{Y} + \frac{k}{4}\left[(D\phi)^2 + (\Delta\phi)^2 + \frac{2}{3}D\phi\Delta\phi - \frac{2}{3}m_0^2\phi^2\right] \tag{20}$$

The first Eq. (20) is exactly the (tr) component of the Einstein field equation (1) in the coordinate form (e.g. Zecca, 2004). The expression of $\psi_2$ is coherent with the analog expression of the Toman-Bondi model (Zecca, 1993, 2000).

To complete the scheme, one must take into account also the scalar field equation (2). To that end, consider that $\nabla_\alpha \nabla^\alpha = \nabla_{A\dot{A}}\nabla^{A\dot{A}}$ and that

$$\nabla_{A\dot{X}}\nabla^{A\dot{X}}\phi = \partial_{A\dot{X}}\partial^{A\dot{X}}\phi + \left[\Gamma^{A}_{A\dot{X}B} + \overline{\Gamma}^{\dot{Y}}_{\dot{Y}B\dot{X}}\right]\partial^{B\dot{X}}\phi \tag{21}$$

where $\nabla_{A\dot{A}}$ and $\partial_{A\dot{X}}$ are the covariant and directional spinorial derivatives (8) respectively. By expliciting the value of $\Gamma^{C}_{A\dot{X}B}$ in terms of the spin coefficients (Chandrasekhar, 1983; Penrose and Rindler, 1984) and taking also into account the result (22), the scalar field equation (2) finally reduces to (compare with Penrose and Rindler, 1984)

$$(\Delta D + D\Delta)\phi + 2(\epsilon - \rho)\,\Delta\phi + 2(\mu - \gamma)D\phi + m_0^2\phi = 0. \tag{22}$$

Therefore, Eqs. (9)–(18) coupled to Eq. (22) remain to be solved.

## 3. TRAPPED SURFACES

The solution of the equations, even simplified as in the previous section, is very difficult. As mentioned in Section 1, the problem has been diffusedly studied and our object remains to give, as far as possible, explicit solutions. To that end, some general considerations about light propagation and trapped surfaces are useful. Suppose the tetrad (4) is subjected to a type III transformation (e.g Chandrasekhar, 1983):

$$\begin{aligned} e_1^\mu &\to \hat{e}_1^\mu = A^{-1}e_1^\mu, \quad e_2^\mu \to \hat{e}_2^\mu = Ae_2^\mu \\ e_3^\mu &\to \hat{e}_3^\mu = e^{i\chi}e_3^\mu, \quad e_4^{\star\mu} \to \hat{e}_4^\mu = e_4^{\star\mu}e^{-i\chi} \end{aligned} \tag{23}$$

the real functions $A$, $\chi$ being defined on the space–time manifold. If the functions are chosen in such a way that the transformed $\epsilon$ spin coefficient vanishes, $A^{-1}\epsilon - 2^{-1}A^{-2}\,DA + 2^{-1}iA^{-1}\,D\xi = 0$ (e.g. $\chi = 0,\ D\log|A| = 2\epsilon,\ A = A(r,t)$), then $\hat{k} = \hat{\epsilon} = \hat{\pi} = \hat{\sigma} = 0$ so that the $\hat{e}_1^\mu$ vectors form a congruence of null geodesic affinely parameterized: $\hat{e}_{1\mu;\nu}\hat{e}_1^\nu = 0$ (Chandrasekhar, 1983; Sachs, 1961). The non-trivial optical scalar is $-\hat{\rho}$ and its variation along a geodesic $g$ of the congruence is

$$\hat{D}\hat{\rho} = \hat{\rho}^2 + \hat{\phi}_{00}. \tag{24}$$

($\hat{\rho} = A^{-1}\rho,\ \hat{D} = A^{-1}D,\ \hat{\phi}_{00} = A^{-2}\phi_{00}$). Further information can be obtained by following the argument considered by Penrose (1968). The area $\mathcal{A}$ of a small triangle in the space-like plane orthogonal to the direction of the geodesic $g$ satisfies the equations

$$\hat{D}\mathcal{A}^{1/2} = -\hat{\rho}\mathcal{A}^{1/2}, \qquad \hat{D}^2\,\mathcal{A}^{1/2} = -\mathcal{A}^{1/2}\,\hat{\phi}_{00} \tag{25}$$

Along the geodesic one has also similar equations for the $Y$ function

$$\hat{D}Y = -\hat{\rho}Y \qquad \hat{D}^2\,Y = -Y\,\hat{\phi}_{00} \tag{26}$$

where the second equation follows from the first one and from Eq. (24) and, without loss of generality, one may consider $Y > 0$.

If $\hat{\rho} > 0$ in some point of the geodesic, then Eq. (25) implies that $\mathcal{A}^{1/2}$ necessarily decreases to zero and the "beam inevitably reaches a focal point $Q$" (Penrose, 1968), thus revealing the existence of trapped surfaces and hence of black hole. Equations (25) and (26) can be integrated along the geodesic and give

$$\hat{\rho} = -\hat{D}\,\log\sqrt{\mathcal{A}} = -\hat{D}\log Y \quad \Rightarrow \quad \mathcal{A} = aY^2 \tag{27}$$

$a$ a constant function along g. If $p$ is the affine parameter of the geodesic, then $Y(p)$ is strictly decreasing and necessarily $Y(p_0) = 0$ for some $p_0$, as a consequence of Eqs. (24), (26) and assumptions on $\hat{\rho}$. On the other hand, $\hat{D}Y(p)$ cannot vanish for $p \to p_0$ because otherwise the concavity of $Y(p)$ would change sign for some $\overline{p} < p_0$, a possibility prevented by the fact that $\hat{D}^2 Y \leq 0$. Therefore, from Eq. (26), $\hat{\rho}(p) \to \infty$ by approaching $Q$. If also $\hat{D}^2 Y(p)$ does not vanish for $p \to p_0$ then also $\hat{\phi}_{00} \to \infty$.

If on the other hand, $\hat{\rho}$ is negative in a point of $g$, $Y(p)$ is increasing in $p$, and, on account of Eq. (26), it may tend to a finite or to an infinite value. Since the function $A$ is determined up to a sign, one has the fact that for every observer for which the light rays focus there exists another observer for which there exist light rays that do not focus, and conversely.

## 4. THE COMPLETELY STATIC CASE

The discussion is now restricted to the completely static case: $\phi = \phi(r)$, $Y = Y(r)$, $\Gamma = \Gamma(r)$. We now show that the solution is possible only if $m_0 = 0$ in which case it is also explicitly determined. By expliciting with respect to the spin coefficients and the directional derivatives, the scalar field equation (22) becomes

$$\phi'' - \left(\frac{\Gamma'}{2} - 2\frac{Y'}{Y}\right)\phi' - e^{\Gamma}\,m^2\,\phi\; = 0 \tag{28}$$

Similarly, by expliciting Eq. (9) one gets

$$-\frac{Y''}{Y} + \frac{\Gamma'}{2}\,\frac{Y'}{Y} = \frac{k}{2}\,\phi'^2 \tag{29}$$

It is possible to obtain another expression of $\phi'^2$ by comparing Eqs. (10) and (12) and expliciting

$$\phi'^2 \;=\; \frac{2}{k}\frac{1}{Y^2}(Y'^2 - e^{\Gamma}) \tag{30}$$

Comparing Eqs. (29) and (30), one obtains a Bernoulli equation (compare with Zecca, 2001)

$$Z' - 2\Big(\frac{Y''}{Y'} + \frac{Y'}{Y}\Big)Z = -2\frac{Z^2}{YY'}\ , \qquad Z = \exp\Gamma, \tag{31}$$

whose solution is given by

$$e^{\Gamma} = \frac{Y^2\,Y'^2}{Y^2 - b^2} \tag{32}$$

$-b^2$ an arbitrary integration constant. Therefore, from Eqs. (30) and (32)

$$\phi' = \pm i b\sqrt{\frac{2}{k}}\frac{Y'}{Y\sqrt{Y^2 - b^2}},$$

$$\phi'' = \phi'\left(\frac{\Gamma'}{2} - 2\frac{Y'}{Y}\right),$$

$$\phi = \phi_0 \pm i\sqrt{\frac{2}{k}}\ \sinh^{-1}\frac{b}{Y}, \qquad (b^2 < 0) \tag{33}$$

(if $b^2 > 0$, the expression of $\phi$ can be obtained from the substitution $\sinh^{-1} \to \sin^{-1}$). By using the expression of $\phi''$ into Eq. (28) it is immediate to see that the solution is not possible if $m_0 \neq 0$, while the scalar field equation is automatically satisfied if $m_0 = 0$. The static case has therefore solution only for massless scalar field that depends on an arbitrary function. In Zecca (2001) it has been shown that arbitrariness cannot be ruled out even by considering non-minimally coupled version of the massless scalar field. The arbitrariness is similar to that of the choice of the radial coordinate and it does not seem to have a particular physical significance. By choosing the radial coordinate to be $Y$ eliminates the arbitrariness and in this case the space–time solution is

$$ds^2 = dt^2 - \frac{Y^2}{Y^2 - b^2}dY^2 - Y^2(d\theta^2 + \sin^2\theta\, d^2\varphi) \tag{34}$$

In correspondence to (34) and (33), one has

$$\rho = -\frac{1}{\sqrt{2}}\frac{Y'}{Y}\,e^{-\Gamma/2} \equiv -\frac{1}{\sqrt{2}}\ \frac{\sqrt{Y^2 - b^2}]}{Y^2} \tag{35}$$

that remains negative where the solution is acceptable. Therefore, according to the considerations of the previous section, the light rays cannot focus.

*Remark* If $b^2 > 0$, one may be tempted to consider the metric (34) in regions $Y^2 < b^2$ where $Y$ is not solution of our problem. These regions have peculiar properties as it follows by studying the null geodesic from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\dot{t}^2 - \frac{Y^2\dot{Y}^2}{Y^2 - b^2} - Y^2\dot{\theta}^2 - Y^2 \sin\theta^2\dot{\varphi}^2\right) \tag{36}$$

(dot means here $d/d\tau$, $\tau$ the proper time). By following Chandrasekhar (1983), one may conclude that the geodesic is described in an invariant plane (say $\theta = \pi/2$) where

$$p_t = E, \qquad p_\varphi = Y^2\,\dot{\varphi}^2 = L$$
$$E^2 = \left(\frac{dY}{d\tau}\right)^2 \frac{Y^2}{Y^2 - b^2} + \frac{L^2}{Y^2} \tag{37}$$

$E$, $L$ integration constants, $L$ the angular momentum about the z-axis. From the last equation, it follows that solutions are possible not only for $Y^2 \geq \max\{b^2, L^2/E^2\}$ but also for $Y^2 \leq \min\{b^2, L^2/E^2\}$. Therefore, in these last regions radial light ray propagation ($L = 0$) is not possible. Instead non-radial light propagation is possible but the light rays do not necessarily collapse in $Y = 0$. Indeed a circular orbit, $Y = Y_0$, $E^2\,Y_0^2 = L^2$, exists that is solution of Eq. (37).

## ACKNOWLEDGEMENT

It is a pleasure to thank the referee for useful comments.

## REFERENCES

Abraham, M. A. and Evans, C. R. (1993). Critical behavior and scaling in vacuum axisymmetric gravitational collapse. *Physical Review Letters* **70**, 2980.

Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes*, Oxford University Press, Oxford.

Choptuik, M. W. (1993). Universality and scaling in gravitational collapse of a massless scalar field. *Physical Review Letters* **70**, 9.

Choptuik, M. W., Hirshman, E. W., Lieblig, S. L., and Pretoriou, F. (2003). Critical collapse of the massless scalar field in axisymmetry. *Physical Review* **D68**, 044007.

Christodoulou, D. (1986). The problem of self-gravitating scalar field. *Communications in Mathematical Physics* **105**, 337.

Christodoulou, D. (1987). A mathematical theory of gravitational collapse. *Communications in Mathematical Physics* **106**, 613.

Evans, C. R. and Coleman, J. S. (1994). Critical phenomena and self-similarity in the gravitational collapse of radiation fluid. **72**, 1782.

Hirshman, E. W. and Eardley, D. M. (1995a). Universal scaling and echoing in the gravitational collapse of a complex scalar field. *Physical Review* **D51**, 4198.

Hirshman, E. W. and Eardley, D. M. (1995b). Critical exponents and stability at the black hole threshold for a complex scalar field. *Physical Review* **D52**, 5850.

Krasinski, A. (1997) *Inhomogeneous Cosmological Models*, Cambridge University Press, Cambridge.

Newman, E. and Penrose, R. (1962). An Approach to Gravitational Radiation by a Method of Spin Coefficients. *Journal of Mathematical Physics* **3**, 566.

Penrose, R. (1968). Structure of space–time. In: de Witt, M. and Wheeler, J. A. (Eds.), *Battelle Rencontres: 1967 Lectures in Mathematical Physics*, Benjamin, New York.

Penrose, R. and Rindler, W. (1984). *Spinors and Space–Time*, Cambridge University Press, Cambridge.

Sachs, R. (1961). Gravitationa waves in general relativity. VI. The outgoing radiation condition. *Proceedings of the Royal Society of London, Series A*, **264**, 309.

Whang, A. (2003). Critical collapse of a cylindrically symmetric scalar field in four-dimensional Einstein's theory of gravity. *Physical Review* **D68**, 064006.

Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, New York.

Zecca, A. (1993). Some remarks on Dirac's equation in the Tolman-Bondi geometry. *International Journal of Theoretical Physisc* **32**, 615.

Zecca, A. (2000). Weyl spinor and solutions of massless free field equation. *International Journal of Theoretical Physisc* **39**, 377.

Zecca, A. (2001). Static solutions of gravitating scalar and matter field in spherically symmetric comoving systems. *Il Nuovo Cimento* **B116**, 169.

Zecca, A. (2003). Self gravitating scalar field in spherically symmetric comoving system: Solutions depending on one variable. *Il Nuovo Cimento* **B118**, 953.

Zecca, A. (2004). Static massless and time dependent massive self gravitating scalar field in spherically symmetric comoving system. *Il Nuovo Cimento* **B119**, 521.